Peter Trapa lectures part 2 at Atlas Conference, Summer 2010, in Salt Lake City, Utah

July 28, 2010

1 Lecture on Cells and Associated Varieties on 07/21/10 at 11AM

Setting: Fix a real group $G(\mathbb{R})$ as in Jeff's talk, and let $K(\mathbb{R})$ be the maximal subgroup. Have $\mathfrak{g} = \mathfrak{g}(\mathbb{R}) \otimes \mathbb{C}$ and $k = k(\mathbb{R}) \otimes \mathbb{C}$. We have the complex groups K, G. We can introduce some interesting invariants attached to $(\mathfrak{g}, K)$-modules, and how to compute them using atlas.

So let X be an irreducible $(\mathfrak{g}, K)$-module. We can consider its annihilator in $U(\mathfrak{g})$, denoted $Ann_{U(\mathfrak{g})}(X) \lhd U(\mathfrak{g})$. The universal enveloping algebra has a grading by degree, so we can take the associated graded ideal $gr(Ann(X))$. Recall that $gr(U(\mathfrak{g})) = S(\mathfrak{g})$, the symmetric algebra, by Poincare-Birkhoff-Witt. We can pass to the variety in $\mathfrak{g}^*$ cut out by $gr(Ann(X))$. We call that the associated variety, $AV(Ann(X)) \subset \mathfrak{g}^*$. Since X is irreducible, $Ann_{Z(\mathfrak{g})}(X)$ is a maximal ideal. By Kostant, $\mathbb{C}[\mathcal{N}(\mathfrak{g}^*)] = S(\mathfrak{g})/gr(Z_0(\mathfrak{g}))$, where $gr(Z_0(\mathfrak{g}))$ is the gr of the augmentation ideal in $Z(\mathfrak{g})$. Therefore, $AV(Ann(X)) \subset \mathcal{N}(\mathfrak{g}^*)$. In fact,

Theorem 1.1. *There exists a unique nilpotent coadjoint orbit $O \subset \mathcal{N}(\mathfrak{g}^*)$ such that $AV(Ann(X)) = \overline{O}$.*

Fact: O has a canonical simplectic structure. We then define $dim(X) := dim(AV(Ann(X))/2$. This is called the *Gelfand-Kirillov dimension* of X.

Let's talk about Cells of Harish-Chandra modules.

Let X, Y be irreducible $(\mathfrak{g}, K)$-modules with the same fixed infinitesimal character λ. Write $X > Y$ if there exists a finite dimensional $(\mathfrak{g}, K)$-module appearing in $T^{\cdot}(\mathfrak{g})$ (tensor algebra) such that Y is a subquotient of $X \otimes F$. Write $X \sim Y$ if $X > Y$ and $Y > X$.

Definition 1.2. $cell(X) := \{Y : Y \sim X\}$
$cone(X) := \{Y : Y < X\}$
$cone^0(X) := \{Y \in cone(X) : Y \notin cell(X)\}$

Assume from now on that we're only working with $(\mathfrak{g}, K)$-modules with infinitesimal character ρ.

Fix $\mathbb{Z}[cone(X)] \subset Groth((\mathfrak{g}, K)$-modules with trivial infinitesimal character), where Groth means Grothendieck group. Recall that W acts on the right hand side. In fact, W also acts on $\mathbb{Z}[cone(X)]$ is also has an action by W, and it is indeed a subrepresentation of $Groth((\mathfrak{g}, K)$-modules with trivial infinitesimal character). $\mathbb{Z}[cone^0(X)] \subset \mathbb{Z}[cone(X)]$ is a further subrepresentation, so W acts on $\mathbb{Z}[cone(X)]/\mathbb{Z}[cone^0(X)] = \mathbb{Z}[cell(X)]$. This is the cell representation of W on $\mathbb{Z}[cell(X)]$.

Proposition 1.3. *If $X \sim Y$, then $AV(Ann(X)) = AV(Ann(Y))$.*

Question: Which nilpotent coadjoint orbits arise as dense orbits in $AV(Ann(X))$, where X has trivial infinitesimal character, for any $G(\mathbb{R})$?

Answer(Barbasch-Vogan): The special orbits (in the sense of Lusztig).

Example: The minimal orbit for $\mathfrak{sp}(4, \mathbb{C})$ is not special.

Let's now explain how to use atlas (and the character table for W) to compute special orbit attached to a cell.

Definition 1.4. A *special W-representation* is one that the Springer correspondence attaches to the trivial local system on a special orbit.

Fix a cell C. $\mathbb{Z}[C]$ contains a unique special Weyl group representation. Atlas computes cells and the wcell command contains the information of the cell representation. It gives integer matrices for action of simple reflections. So use the character table to write as a sum of irreducibles of irreducible W-representations. Using tabulation of special W-representations, locate the unique special constituent. Using tables for Springer correspondence, get the special orbit parameterizing $AV(Ann(X))$, $X \in cell$.

2 Lecture on Cells and Associated Varieties on 07/22/10 at 11AM

Last time: We had $K(\mathbb{R}) \subset G(\mathbb{R})$. If X is an irreducible HC-module, we had some interesting algebraic invariants: $Ann_{U(\mathfrak{g})}(X), AV(Ann(X)), dim(X) := \frac{1}{2}dim(AV(Ann(X)))$.

Let's talk about a different invariant. Recall that $AV(Ann(X)) = \overline{O}$ where O is a coadjoint orbit in $\mathcal{N}(\mathfrak{g}^*)$. We want to now consider something called $AV(X)$, which is a subset of $\mathcal{N}(\mathfrak{g}^*) \cap (\mathfrak{g}/\mathfrak{k})^* =: \mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.

Here's how to define this. Let X_0 be a finite dimensional K-invariant subspace of X which generates X. Define $X_i := U^{\leq i}(\mathfrak{g})$, where $\leq i$ means the less than or equal to i degree terms in the universal enveloping algebra. So we have a K-invariant, exhaustive filtration on X:

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

$gr(X_i)$ is a $grU(\mathfrak{g}) = S(\mathfrak{g})$-module. Not hard to see that it descends to a $S(\mathfrak{g}/\mathfrak{k})$-module.

Definition 2.1. Define $AV(X) := \{\lambda \in (\mathfrak{g}/\mathfrak{k})^* : p(\lambda) = 0 \ \forall p \in Ann_{S(\mathfrak{g}/\mathfrak{k})}(gr(X))\} \subset \mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.

Since X has infinitesimal character, $AV(X) \subset \mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$. Moreover, $AV(X)$ is a K-invariant subset of $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.

Theorem 2.2. *(Kostant-Rallis) There are finitely many K-orbits on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.*

Therefore, $AV(X) = \overline{O_K^1} \cup \cdots \cup \overline{O_K^j}$, some K-orbits O_K^i on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.

Theorem 2.3. *(Vogan) Each O_K^i appearing in $AV(X)$ is a Lagrangian subvariety of $O \subset \mathcal{N}(\mathfrak{g}^*)$ where $AV(Ann(X)) = \overline{O}$. (So the Gelfand-Kirillov dimension of X is equal to $dim(AV(X))$.)*

Proposition 2.4. *If $X \sim_{cell} Y$, then $AV(X) = AV(Y)$.*

The only "easy" calculation of AV is as follows: Suppose X is attached to a trivial local system on a K-orbit Q. Suppose that $\overline{Q}$ is smooth. Then write $Q = K \cdot \mathfrak{b}$, where $\mathfrak{b}$ is a representative. Then we have:

Proposition 2.5.

$$\begin{aligned} AV(X) &= K \cdot ((\mathfrak{g}/\mathfrak{b})^* \cap (\mathfrak{g}/\mathfrak{k})^*) \\ &= K \cdot (\mathfrak{g}/\mathfrak{b} + \mathfrak{k})^* \\ &= \overline{O_K} \end{aligned}$$

Example: If X is a discrete series, then X is attached to the trivial local system on an orbit Q such that $Q = \overline{Q}$.

Corollary 2.6. *The associated varieties of discrete series are "known".*

Example: Suppose Q' is a closed K-orbit on a partial flag variety G/P. Write $\pi : G/B \to G/P$. Then there is a unique dense K-orbit in $\pi^{-1}(Q')$, call it Q. And $\overline{Q}$ is smooth. Some representations attached to such Q and the trivial local system are given in output of blocku in atlas. (These are the so-called $A_{\mathfrak{q}}(\lambda)$-modules, unitary,...)

Observation(Barbasch-Vogan): For $G = U(p, q)$, every X with trivial infinitesimal character is a cell containing something from blocku output. We conclude that prop + cell invariane compute AV's for $U(p, q)$.

atlas problem: Use this trick for $Sp(p, q)$. This observation fails. Monty + Trapa listed all orbits for $Sp(p, q)$ with smooth closure.

Question: Does every cell for $Sp(p, q)$ contain X attached to the trivial local system and Q such that $\overline{Q}$ is smooth? If yes, you would have computed all AV's for $Sp(p, q)$.

In an AJM paper, Monty McGovern showed that cells for $Sp(p, q)$ are parameterized by K-orbits on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$.